Weighted Polya Theorem. Solitaire

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Cosets

- For a group G and a subgroup $H \subset G$, cosets are subsets of G of the form gH and Hg for $g \in G$.
- Let G act on a set X, pick a point $x \in X$ and let Gx and G_x be its orbit and stabilizer.

  Lemma 1. The orbit Gx is in a natural bijection with the set of cosets $G/G_x = \{gG_x \mid g \in G\}$. In particular, for finite groups, $|Gx| = |G|/|G_x|$.

  Lemma 2. For any other point $y \in Gx$ of the orbit of x, the stabilizer of G_y is $G_y = gG_xg^{-1}$ for some $g \in G$. In particular, for finite groups, all the stabilizers of points from the same orbit have the same number of elements.

Polya's Enumeration Theorem

Theorem

Suppose that a finite group G acts on a finite set X. Then the number of colorings of X in n colors inequivalent under the action of G is

$$N(n) = \frac{1}{|G|} \sum_{g \in G} n^{c(g)}$$

where $c(g)$ is the number of cycles of g as a permutation of X.

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- Let X_n be the set of colorings of X in n colors. Then we want to compute the number of G-orbits on X_n.
- Let's instead count the pairs (g, C) with $C \in X_n$ a coloring and $g \in G_C \subset G$ an element of G preserving C.

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- Let's instead count the pairs (g, C) with $C \in X_n$ a coloring and $g \in G_C \subset G$ an element of G preserving C.
- The orbit GC of C has $|G|/|G_C|$ elements (**used Lemma 1**).
- Each element of GC will appear $|G_C|$ times (**used Lemma 2**).
- Thus each orbit of X_n will appear $|G_C| \cdot |G|/|G_C| = |G|$ many times in our counting. So to find $N(n)$ need to divide the result by $|G|$.

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- This gives $n^{c(g)}$ invariant colorings.
- Summing over all $g \in G$ and dividing by $|G|$ gives the required formula.

Weighted Polya theorem

Let $c_m(g)$ denote the number of cycles of length m in $g \in G$ when permuting a finite set X.

Theorem (Weighted Polya theorem)

The number of colorings of X into n colors with exactly r_i occurrences of the i-th color is the coefficient of $t_1^{r_1} \dots t_n^{r_n}$ in the polynomial

$$P(t_1, \dots, t_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{m \geq 1} (t_1^m + \dots + t_n^m)^{c_m(g)}$$

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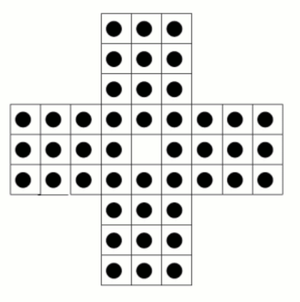
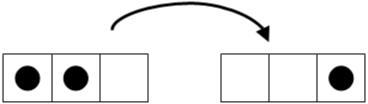
- What is the number of necklaces with exactly 2 white and 2 black beads? exactly 1 white and 3 black?

(Peg) Solitaire board

Solitaire rules



A move in the game consists of picking up a marble, and jumping it horizontally or vertically (but not diagonally) over a single marble into a vacant hole, removing the marble that was jumped over.



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- The game is won by finishing with a single marble left on the board, in the central hole.
- **Question:** is it easier to win the game finishing at **any** spot on the board?
- In other words, are there more winning strategies if we relax the winning condition?
- Color spots on the board with **non-trivial** elements of $\mathbb{Z}/2 \times \mathbb{Z}/2$ so that for any 3 consecutive positions (row or column) there are all three elements (let's call them f, g, h).

  (We just re-denote $f = (1,0), g = (0,1), h = (1,1)$.)

Filled board

			f	g	h			
			g	h	f			
			h	f	g			
f	g	h	f	g	h	f	g	h
g	h	f	g	h	f	g	h	f
h	f	g	h	f	g	h	f	g
			f	g	h			
			g	h	f			
			h	f	g			

Main trick

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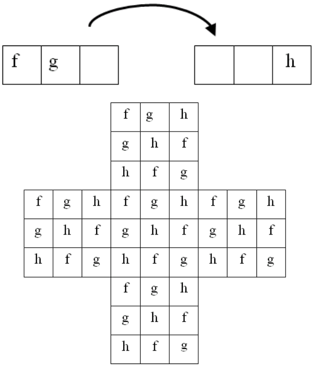
- The value of the board does not change when doing moves!



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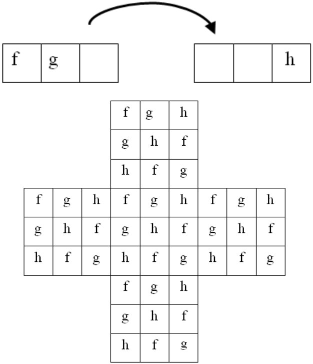

- So we should end up with a marble in a position labeled by h (15 possibilities).

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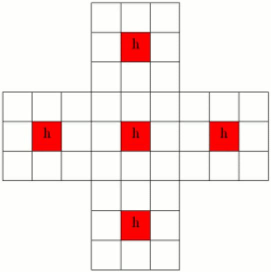
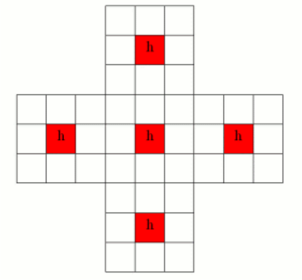
- Thus, if there is a sequence of moves finishing in one spot, then there is a sequence of moves finishing in a symmetric spot.
- In other words, there is an action of the group D_4 on the set of all possible states of the board.
- Thus we can only finish in the following spots:



The end



- If we finished the game in one of the 4 non-central positions. How could that happen?

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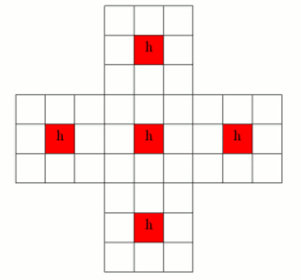

- If we finished the game in one of the 4 non-central positions. How could that happen?
- So we might have as well finished in the middle spot.

Generalizations

What about Solitaire games of other shapes?

Figure: French Solitaire

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